Lecture 1:

We will develop the theory of complex differentiable functions (we will see soon what this means), and it is surprisingly elegant. Examples of complex differentiable functions are complex polynomials $a_0 + a_1 z + a_2 z^2 + \cdots + a_d z^d$. Other examples are functions defined by infinite power series such as $a_0 + a_1 z + a_2 z^2 + \cdots$, or functions like $\zeta(s) = \sum_{n=0}^{\infty} \frac{1}{n^s}$ for Re(s)>1. Also, we will study a connection to harmonic functions: if we have a disk in the plane and a twice real differentiable function u of x and y, then we say u is harmonic if $u_{xx} + u_{yy} = 0$.

Notation: z will denote a complex variable and we will write x for the real part and y for the imaginary part. We will write $\mathbb{C}^*$ to mean $\mathbb{C}\backslash 0$. Recall that a set U in $\mathbb{C}$ is open if it contains an open disc about each of its points.

Definition: A path in U∈ℂ or ℂ is a continuous map γ from a closed interval [0,1] to ℂ. We say U is path connected if any two points in U are joined by a path.

Definition: A domain in ℂ is a non-empty open path connected set.

Definition: We say γ is a closed path if $\gamma(0) = \gamma(1)$. When we integrate along paths we will need these paths to be continuously differentiable. We say γ is C^n if it is n times continuously differentiable everywhere in [0,1]. Often, paths are always continuous but only piecewise C^1 and these are the paths we will mainly work with. We say a path is simple if it is injective except for perhaps $\gamma(0) = \gamma(1)$.

Definition: Let U∈ℂ be open. We say $f: U \to \mathbb{C}$ is differentiable at w if the analog of normal differentiability holds, ie $\lim_{z\to w} \frac{f(z)-f(w)}{z-w}$ exists. We say f is holomorphic at w if it is differentiable on some open neighbourhood of w. If $U = \mathbb{C}$ and $f: \mathbb{C} \to \mathbb{C}$ is holomorphic everywhere then we say that f is an entire function. Sometimes this is called being analytic.

Note that this definition is different from real differentiability (we will see how soon).

Note that the usual rules for differentiating sums, products, compositions, and inverses are the same, because you can use exactly the same proofs as in the real case.

Any function $U \to \mathbb{C}$ can be written as $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ for some real valued functions u and v that are just the real and imaginary parts of f. Recall that $u: U \to \mathbb{R}$ is differentiable at $(c,d) \in U \in \mathbb{R}^2$ if and only if there are λ,μ such that $\lim_{(x,y)\to(c,d)} \frac{u(x,y)+u(c,d)-\lambda(x-c)-\mu(y-d)}{\sqrt{(x-c)^2+(y-d)^2}} = 0$.

Proposition (Cauchy Riemann equations):

$f(x+iy) = u(x,y) + iv(x,y)$ is differentiable at (c,d) with derivative p+iq if and only if u and v are both differentiable there and $u_x = v_y = p$ and $-u_y = v_x = q$.

Proof:

Suppose first that f is differentiable at w with derivative p+iq, then $\lim_{z\to w} \frac{f(z)-f(w)-(z-w)(p+iq)}{z-w} = 0$. We use the simple fact that $\lim_{z\to a} \frac{f(z)}{g(z)} = 0$ if and only if $\lim_{z\to a} \frac{f(z)}{|g(z)|} = 0$. If z=(x,y) and w=(c,d) then by simple algebra

$$(z-w)(p+iq) = p(x-c) - q(y-d) + i\big(q(x-c) + p(y-d)\big)$$

We know the real and imaginary parts to both go to 0 if the limit goes to 0, so (using the facts above)

$$\lim_{z\to w}\frac{u(x,y)-u(c,d)-\big(p(x-c)-q(y-d)\big)}{\sqrt{(x-c)^2+(y-d)^2}}=0$$

$$\lim_{z\to w}\frac{v(x,y)-v(c,d)-\big(q(x-c)+p(y-d)\big)}{\sqrt{(x-c)^2+(y-d)^2}}=0$$

So u and v are differentiable at (c,d) with $Du=(p,-q), Dv=(q,p)$. This proves the first direction.

Remark before we prove the other direction: We don't merely need $u_x=v_y=p$ and $-u_y=v_x=q$ but that u and v are differentiable as functions $\mathbb{R}^2\to\mathbb{R}$. However, if u and v have continuous partials in an open neighbourhood of w then we are good.

The converse is then immediate from the idea that limit in complex numbers is equivalent to limit in real and imaginary parts.

Note that polynomials are entire – we can derive the limit the same way we can for real functions. Ie, polynomials are sums and products of the identity function and constants which are clearly entire.

Example of something that does not work: $f(z)=|z|$ is differentiable in the real sense at everywhere but 0, but nowhere in the complex sense. This is because $u=\sqrt{x^2+y^2}, v=0$, and we compute directly that $u_x=\frac{x}{\sqrt{x^2+y^2}}, u_y=\frac{y}{\sqrt{x^2+y^2}}, v_x=v_y=0$ so the cauchy riemann equations fail, but at the origin this is not differentiable either as the limit is 1 along the positive reals and -1 along the negative reals.

Lecture 2:

Suppose u and v are twice continuously differentiable and satisfy the cauchy riemann equations, then $u_x=v_y$ so $u_{xx}=v_{yx}=v_{xy}=-u_{yy}$ so $u_{xx}+u_{yy}=0$.

Proposition: Let U be a domain and f be holomorphic, then if f'(z)=0 everywhere on U then f is constant.

Proof: Suppose we have a continuous path, then since U is in an open region at any point we can cover it by an open rectangle contained in U such that its boundary is contained in U, and since the path is compact we can extract a finite subcover. We can use the edge of these rectangles to construct an axis-aligned path from a to b, as we did in differential equations for that one proof in Lecture 8.

Now on this path, $u_x=v_y=u_y=v_x=0$, so by the known real analysis version of this theorem, the derivatives vanish on u so u and v are constant along this path, thus f is constant along this path, so f(a)=f(b), then since a and b were arbitrary we are done.

Definition: We say f is conformal at w if it is holomorphic with non-zero derivative.

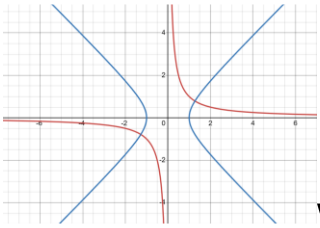
We now consider the image of a path γ_1 that locally looks like $(-\varepsilon,\varepsilon)$ and a path γ_2 that locally looks like $(-\varepsilon,\varepsilon)e^{i\theta}$ under a function conformal at 0, ie they have non-zero tangent vectors with an angle θ between them.

Now $\frac{f(\gamma_1(0))}{f(\gamma_2(0))} = \frac{f'(0)}{f'(0)}\frac{\gamma_1'(0)}{\gamma_2'(0)}$ by the chain rule. This means that the image of two paths that intersect under a function which is conformal at their point of intersection preserves the angle between the paths. The intuition for this is that real differentiability needs the limit to exist from the left and the right, but complex differentiability needs it to exist in all directions which is a much stronger condition. We say f is a conformal equivalence if it is a bijection from domains U to V and conformal on all of U. The bijection property + the inverse function theorem viewing $\mathbb{C}$ as $\mathbb{R}^2$ and in this context one can see that non-zero complex derivative is equivalent to non-singular derivative, you can see this if you use the cauchy riemann equations and see that a p+qi derivative corresponds to the derivative matrix $\begin{pmatrix} p & q \\ q & -p \end{pmatrix}$, where this matrix geometrically is the same as multiplying the complex plane by p+qi.

Conformal mappings with a continuous derivative are guranteed to be local bijections by the inverse function theorem. We show later that if a function is holomorphic on a domain then its derivative has to be continuous and we will be able to drop this extra assumption.

Example:

If $f(z) = z^2$ then $f(x+iy) = x^2 - y^2 + 2xyi$. By earlier, it must be the case that the pre-image of any two perpendicular lines not through the origin are perpendicular. What this means is that we can deduce that the graphs $x^2 - y^2 = c$ and $2xy = c'$ or $y = \frac{c'}{x}$ are always perpendicular for ANY c and c' as long as it is not the case that both of them are 0.


We see in this image that the graphs do indeed meet at right angles.

Example: If a is not 0 then az+b is a conformal equivalence $\mathbb{C}$ to $\mathbb{C}$. $f(z) = z^n$ takes the points satisfying $0 < \arg(z) < \frac{\pi}{n}$ to the points with strictly positive imaginary part and is a conformal equivalence. Note that this "upper half plane" is exactly the points that are closer to i than -i, which means $\left|\frac{z-i}{z+i}\right| < 1$. Therefore this mobius map takes the upper half plane to the unit circle in the complex plane. This is a conformal equivalence since the derivative is $\frac{2i}{(z+i)^2}$ which is not 0 in the upper half plane, and it is clearly injective and surjective.

Example: The exponential map sends vertical lines in the plane to circles centered at the origin whose radius is determined by the real part. This is not in general conformal since it is not always a bijection and the resulting circle will wrap around the origin infinitely often. Other lines are sent to spirals or half-lines (in the horizontal case).

Example: From the groups course, it is known that mobius maps send lines and circles to lines and circles. So the image of a sector a<arg(z)<b under a mobius map is a rgion bounded by two lines or circles (since mobius maps are bijective). So we have a few possibilities:

1. We get another sector if the mobius map sends lines to lines, ie it is rotation+stretching+shifting but with no division.

2. The two lines that define the sector (which intersect at 0 and infinity) end up intersecting at two places in the plane in a fashion that we get a line and a circle, and any of the regions deliniated by these lines can be the image.
3. Same as case 2 except we have 2 circles.

In these cases we can determine which is the image by considering where some point in the sector goes.

Example: The map $\frac{z^2+1}{2z}$ is holomorphic except at 0 and conformal except at 0, 1 or -1 (differentiate this to check).

Lecture 3:

Let S^1 be the unit circle (the boundary of the closed unit disc).

Let U be a domain. We say U is simply connected if every continuous map from the circle to the domain extends to a continuous map of the closed disc. This intuitively means that any loop can be continuously shrunk to a point so there are no holes

Recall from level 4 that if f(z) is defined by a complex power series defined by $\sum c_n(z-a)^n$ with non-zero radius of convergence R then its derivative is given termwise and has the same radius of convergence. Therefore f(z) is holomorphic and has holomorphic derivative (and in fact infinitely many holomorphic derivatives) in $B_R(a)$.

Proposition: Suppose that a power series has radius of convergence R about a and suppose that on a ball of radius ε about a the power series is 0, then it is always 0.

Proof: All derivatives are 0 at a (known) so all coefficients are 0. This is because we know from level 4 that differentiation of power series is given by termwise differentiation of the coefficients.

Recall that the exponential function hits every complex value except for 0.

We will talk about the logarithm. Since the exponential is "many-to-one", the logarithm "wants" to be one-to-many.

If U is a domain not including 0, then a continuous function F on U is a branch of the logarithm is everywhere on U $e^{F(z)} = z$.

Now if U is within the complex plane cutting out 0 and the negative reals, the principle value of the natural logarithm on here (as discussed in level 4) is a branch of the logarithm that works.

The problem is if our domain wraps around the origin, and we try to make the imaginary part of the log vary continuously, it will increase by $\pm 2\pi$ by the time we get back to where we started, so we cannot have a single valued logarithm on here.

The same problem happens for non-integer powers z^n since they are defined as $e^{nlog(z)}$. If n is an integer this is fine because that is the only case where log changing by $2i\pi$ does not change the result.

Lecture 4:

Note that on any simply connected domain not containing 0, a single valued branch of the logarithm exists on this (even if it spirals around the origin), since as we will show later in the course, any function which is holomorphic on a simply connected domain (and in particular $\frac{1}{x}$) has a single valued antiderivative on there.

We note that if n is rational in $f(z) = z^n$ then this is multivalued but only has finitely many values, since it is some finite integer root of a single valued function, and finite integer roots are finite valued in the complex numbers.

Since $\sqrt{z} = e^{\frac{1}{2}\log(z)}$, then it is clear that we cannot really define a global square root function since when we go around the origin and try to vary it continuously we will end up going from 1 to -1. See the proof of unsolvability of the quintic in the misc results section of this website for more details on this.

Lets try to look at the function $f(z) = \sqrt{z(z-1)}$. Then it turns our that if we slit the plane by removing $[0,1]$ or $(-\infty, 0] \cup [1, \infty)$ we can get a single valued continuous function.

We can think of $(-\infty, 0] \cup [1, \infty)$ as one line through the point at infinity rather than two different lines.

$\sqrt{z(z-1)} = e^{\frac{1}{2}\log(z(z-1))}$. Now informally, if we traverse a loop that avoids $[0,1]$, the argument of $\log(z)$ and $\log(z-1)$ both try to jump by 2π so $\frac{1}{2}\log\big(z(z-1)\big)$ tries to jump by 2π and not π so we don't have the same problem.

Definition: A point p in $\mathbb{C}$ is a branch point of a multi valued function if the function cannot be defined in a continuous single valued manner on any punctured neighbourhood around the point. An example is 0 for log or z^a for any a not an integer. This is exactly like the "bad points" in the unsolvability of the quintic proof in the misc results section.

We say $f: \mathbb{R} \to \mathbb{C}$ is riemann integrable if its real and imaginary parts are. Since we are talking about continuous functions we will not worry about integrability.

Intuitively obvious proposition: $\left|\int_a^b f(t)dt\right| \le \sup_{t\in[a,b]} (b-a)|f(t)|$

Proof: Let $\theta = Arg\left(\int_a^b f(t)dt\right)$ and $M = \sup_{t\in[a,b]} |f(t)|$. Now $\left|\int_a^b f(t)dt\right| = e^{-i\theta}\int_a^b f(t)dt$. But this is just equal to $\int_a^b Re\big(f(t)e^{-i\theta}\big)dt$ since the whole thing is real so the integral of the imaginary part is 0. Now by real integral properties, this is $\le \int_a^b \left|\big(f(t)e^{-i\theta}\big)\right|dt = \int_a^b \left|\big(f(t)\big)\right|dt \le M|b-a|$ with equality if and only if f is constant.

Definition: A contour is any piecewise continuously differentiable closed path $[a,b] \to \mathbb{C}$. We will only integrate along contours and not general continuous curves since those could be fractal curves or space filling curves. Piecewise is fine because we can just add up the integrals of finitely many parts.

Definition: Suppose we have a continuously differentiable path $\gamma: [a,b] \to \mathbb{C}$ (Not necessarily closed). Then we have the intuitive definition $\int_\gamma f(z)dz = \int_a^b f\big(\gamma(t)\big)\gamma'(t)dt$. We will now formalize the stuff we did with complex integrals in levels 4 and 6.

Note that if we pick a different parametrization for the same path, ie $\phi: [a', b'] \to [a,b]$ is a continuously differentiable function which sends a' to a and b' to b, then we consider $\delta = \gamma \circ \phi$, then we have that $\int_\gamma f(z)dz = \int_\delta f(z)dz$. This can be proven by expanding it using the definition and then doing simple integration by substitution.

This only holds if we really are just integrating a function along a path. If we do an integral to find the length like $\int_a^b |\gamma'(t)|dt$ then this is only invariant under re-parametrization if the reparametrization is monotone and does not go back on itself or else the length would be double counted in places.

Now consider integrating z^n along the unit circle C anticlockwise

Then $\int_C z^n dz$ is 0 if n is an integer except if n is -1 in which case it is $2i\pi$.

Proof:

Here our γ is e^{it}.

$$\int_0^{2\pi} e^{nit}(ie^{it})dt = i\int_0^{2\pi} e^{(n+1)it}dt$$

And now it is clear why the formula holds.

We also have the fundamental theorem of calculus (Complex version). This is obvious but I will state it.

If U is a domain and f is holomorphic on U and its derivative is continuous on U, then

$$\int_{\gamma(a)}^{\gamma(b)} f'(z)dz = f\big(\gamma(b)\big) - f\big(\gamma(a)\big)$$

Again, to prove it just plug in the definition and use the real version of the fundamental theorem of calculus. Crucially, if γ is closed we get 0.

Of course, it now follows that if we integrate z^n with n not -1 around any loop, then since it has a global single valued antiderivative except possibly at 0 we will get 0. If we integrate z^{-1} then the result depends on whether we enclose the origin. Around the origin we cannot define an antiderivative – this would contradict the fact that we get 2πi and not 0.

An antiderivative has the obvious definition (holomorphic with derivative f). We want it to not be multivalued like log as an antiderivative of $\frac{1}{z}$.

Lecture 5:

Obvious proposition: If f is continuous on a continuously differentiable path γ then

$$\left|\int_\gamma f(z)dz\right| \leq length(\gamma)\sup_\gamma |f|$$

This is by basic properties of real integration: It is bounded by

$$\int_a^b \big|f\big(\gamma(t)\big)\big||\gamma'(t)|dt \leq \sup\big|f\big(\gamma(t)\big)\big|\int_a^b |\gamma'(t)|dt$$

So the result follows.

Proposition: Let U be a domain and f be continuous on U. Assume that on U, the integral of f is 0 on any contour. Then f has an antiderivative. Note that the converse of this proposition is something which we already know to be true.

Proof:

Pick any point $a_0 \in U$. From earlier, path connectedness and openness implies that any 2 points are connected by a path that is piecewise C^1. So for any w, pick such a path from a_0 to w and integrate along it. Note that since the integral along a loop is 0, this integral is indpendent of our choice of the path (take one path and minus the other path and we must get 0 so the integral along the two paths must be the same). Therefore $\int_{a_0}^{w} f(z)dz$ is a valid antiderivative. We just need to prove the obvious statement that the derivative of this is actually f. Call this "antiderivative" F.

By openness, there is some ε such that $B_\varepsilon(w)$ is in U. Therefore for any complex number h with |h|<ε we can consider the straight path from w to w+h. Call this path δ_h.

Then $F(w+h) = F(w) + \int_{\delta_h} f(z)dz$.

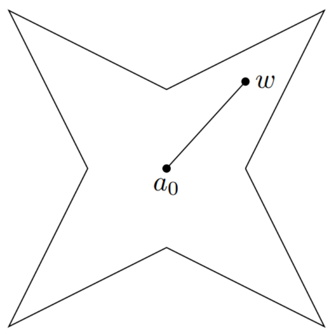
The second term can be written as $\int_{\delta_h} f(z) - f(w)dz + hf(w)$.

Now $\left|\frac{F(w+h)-F(w)}{h} - f(w)\right| = \left|\frac{1}{h}\int_{\delta_h} f(z) - f(w)dz + f(w) - f(w)\right| = \left|\frac{1}{h}\int_{\delta_h} f(z) - f(w)dz\right|$

Note that continuity of f implies that the last term approaches 0 in the limit (by the previous obvious proposition, the mean value of the integral can be made arbitrarily small by continuity), so we have the derivative we want.

Definition: A domain U is convex if for all p and q in U they are connected by a straight line.

Definition: A domain U is star shaped if there exists a point p in U that can be connected to any other point q in U by a straight line. This is weaker than being convex because we just need such a point p to exist, not for every point to be such a point p. This is, however, stronger than being path connected.



Example: The shape in this image is star shaped by a_0 but not convex as a point near two corners cannot be connected by a line.

Proposition: Let U be a star shaped domain and f be continuous on U. Suppose that $\int_\gamma f = 0$ whenever γ is a triangle in U, and the entire interior of the triangle is contained in U, not just the vertices and the edges, then f has an antiderivative.

Proof: Let p be a point that U is star shaped with respect to p. Let γ_w be the straight path from p to w. Note that the paths γ_w, γ_{w+h} and δ_h form a triangle in U, then the rest of the proof proceeds as in the previous proposition.

Now this gets to the exciting bit.

Theorem (Cauchy's theorem for a triangle): Let U be a domain and T in U be a triangle in U. Then if f is holomorphic on U, $\int_T f = 0$. So by the previous theorem this implies any holomorphic function on a star shaped domain has an antiderivative.

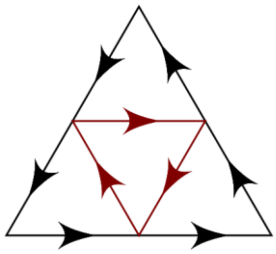
We will suppose the triangle is oriented anticlockwise.

We really do need the function to be defined everywhere inside the triangle – we use this in our proof when we start subdividing the triangle.

Proof:

Let L be the length of the perimeter of the triangle. Let I be $\left|\int_T f\right|$.

Cut T into 4 smaller triangles by bisecting edges like this:



Note that the middle one has to be oriented in the opposite way from the others. It is easy to seee why. Note however that the sum of the integrals over all 4 triangles anticlockwise is the integral of the main triangle since the integral along the interior edges cancel. Therefore there exists a triangle in these 4 which we will call T_1 such that $\left|\int_{\partial T_1} f\right| \geq \frac{I}{4}$. We can construct a nested sequence of triangles lying inside one another this way such that $\left|\int_{\partial T_n} f\right| \geq \frac{I}{4^n}$.

Note that each time we were bisecting edges, so the length of T_n is $2^{-n}L$.

Now we will claim that the intersection of all $T_n's$ is non-empty and contains a unique point. This is intuitive but we must prove it.

Note that the diameter of the triangles approach 0 so there can be at most one point since the points get arbitrary close together in our triangles. The nested intervals property from Analysis I applied to each coordinate implies there is an intersection point.

Define a new function $g(z) = \frac{f(z)-f(w)}{z-w} - f'(w)$ if z is not w and 0 otherwise. Then this is continuous on U by holomorphicity of f. Now $\frac{I}{4^n} \leq \left|\int_{\partial T_n} f\right| = \left|\int_{\partial T_n} f(z) - f(w) - (z-w)f'(w)dz\right|$. The last equality is because the function $f(w) - (z-w)f'(w)$ has an antiderivative (it's just a linear function of z)

Therefore $\frac{I}{4^n} \leq \frac{L}{2^n} \sup_{z\in\partial T_n} |(z-w)g(z)| \leq \frac{L}{2^n} Diam(T_n) \sup_{z\in\partial T_n} |g(z)| = \frac{CL^2}{4^n} \sup_{z\in\partial T_n} |g(z)|$

Where C is the ratio between L and $Diam(T_0)$ which is less than 1 but we just need the fact that it exists and I'm too lazy to show it's actually less than 1. Multiplying both sides of the inequality by 4^n gives that

$$I \leq CL^2 \sup_{z\in\partial T_n} |g(z)|$$

I is therefore 0 by continuity of g since $\sup_{z\in\partial T_n} |g(z)|$ gets arbitrarily close to 0 as z eventually gets sufficiently close to w. So I is arbitrarily small, hence 0, and CL^2 is just a constant.

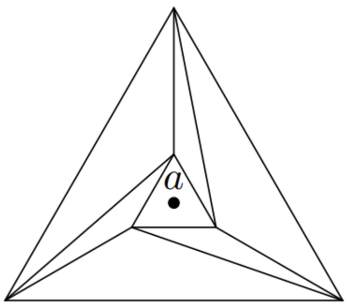
It turns out that a holomorphic function on any simply connected domain has an antiderivative but right now we only have this for star shaped domains. We will come back to this at the end of the course – the result is implied by Cauchy's residue theorem.

Lecture 6:

To recap what we know so far: Having an antiderivative is equivalent to path independence. Holomorphic implies the derivative vanishes on any triangle which in turn implies on star shaped domains that there is an antiderivative.

Proposition: Suppose U is a domain and $f: U \to \mathbb{C}$ is continuous and holomorphic everywhere except for on a finite set. Then the cauchy theorem for triangles still holds. We will show that in reality f must also be holomorphic on those points if these hypotheses holds, but the proof still works even if this is not the case.

Proof: We can suppose that S is just a single point by subdividing T if necessary. If the triangle did not contain any of the points in S we can just shrink U and we are fine. Otherwise we can subdivide the triangle in such a way that there is only one bad point in the triangle, possibly on its boundary.



We can subdivide like this, then the integral along the outer triangle is the same as the integral along the inner triangle, since the intgral along the other triangles is 0. The integral along all the other triangles anticlockwise is 0 but the total is the difference between the integral of the inner and outer triangles. However, we can take the little triangle to be as little as we like, so the integral can be as small as we like by continuity of f, and therefore the whole integral must be 0.

Corollary: If U is a star shaped domain and f is continuous everywhere and holomorphic on all but finitely many points then it has an antiderivative

Theorem: Let U be a domain and f be holomorphic on U. Suppose that $\overline{B_r(a)}$ is in U, then for all z in $B_r(a)$ with $0 \leq |z-a| < r$ we have that $f(z) = \frac{1}{2\pi i}\int_{|w-a|=r} \frac{f(w)}{w-z} dw$.

Proof: Let $g(z) = \frac{f(z)-f(w)}{z-w}$ if w is not z and $f'(z)$ otherwise. Then this is clearly continuous on $\overline{B_r(a)}$ and holomorphic except perhaps at z. So by the previous theorem $\int_{|w-a|=r} g(w)dw = 0$ which implies that $\int_{|w-a|=r} \frac{f(w)}{w-z} dw = \int_{|w-a|=r} \frac{f(z)}{w-z} dw$. Now by simple algebra $\frac{1}{w-z} = \frac{1}{w-a}\frac{1}{1-\left(\frac{z-a}{w-a}\right)} = \frac{1}{w-a}\sum_{n=0}^{\infty}\left(\frac{z-a}{w-a}\right)^n$ which is ok since we assumed that $\left|\frac{z-a}{w-a}\right| < 1$. Hence we have

$$\int_{|w-a|=r} \frac{f(w)}{w-z} dw = f(z) \int_{|w-a|=r} \frac{1}{w-a} \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a}\right)^n dw$$

By the dominated convergence theorem (it is clear that the hypotheses for that theorem hold by the extreme value theorem so we have boundedness), or by uniform convergence of power series in any disc strictly smaller than their radius of convergence (which is true because convergence must be at least as fast as the absolute convergence rate at the edge) and the fact that uniform convergence allows limit and integral swaps for integrals over finite arc lengths. As a reminder, the first part is because we know that everywhere the rate of convergence is at least the rate of absolute convergence (think triangle inequality) which is at least the rate of absolute convergence at the edge of the disc which is something fixed, and the second part is because eventually by uniform convergence every value of the function is less than $\frac{\varepsilon}{path\ length}$ away from the limit which means the integral gets within ε of the integral of the limit.

Anyway, we can swap sum and integral to get

$$\int_{|w-a|=r} \frac{f(w)}{w-z} dw = f(z) \sum_{n=0}^{\infty} \int_{|w-a|=r} \frac{1}{w-a} \left(\frac{z-a}{w-a}\right)^n dw$$

But since z-a is a constant, we know that only the n=0 term survives since the integral of any power of w-a other than -1 goes to 0. We therefore get

$$\int_{|w-a|=r} \frac{f(w)}{w-z} dw = f(z) \int_{|w-a|=r} \frac{1}{w-a} dw$$

And this integral on the right gives us exactly $2\pi i$. This completes the proof.

This result is extraordinary because it lets you determine the value of a function EVERYWHERE just given what it does on the boundary. All just because you can differentiate it.

Lemma: Let U be a domain and f be holomorphic on U. Suppose that $B_r(a)$ is in U, then we have that $f(a) = \int_0^1 f(a + re^{2\pi it}) dt$. The proof is that we can say $f(a) = \frac{1}{2\pi i} \int_{|w-a|=r} \frac{f(w)}{w-a} dw$ by the previous theorem, then parametrizing the path by $\gamma = a + re^{2\pi it}$, we rewrite the integral as

$$f(a) = \frac{1}{2\pi i} \int_0^1 \frac{f(\gamma(t))}{\gamma(t) - a} \gamma'(t) dt = \frac{1}{2\pi i} \int_0^1 \frac{f(a + re^{2\pi it})}{re^{2i\pi t}} 2\pi i r e^{2i\pi t} dt$$

And the result follows.

Corollary (Local maximum principle): A holomorphic function on an open ball cannot achieve its maximum at the center, otherwise it is constant.

Proof: $|f(a)| = \left|\int_0^1 f(a + re^{2\pi it}) dt\right|$ which is less than the supremum of the function on the boundary. Equality only holds if the function is equal to f(a) everywhere on the circle, and since this holds for all r f must be constant on the disc.

It follows that a holomorphic function cannot have a local maximum in an open domain.

Now it just keeps getting better.

Corrolary (Liouville's Theorem): Let f be holomorphic everywhere in $\mathbb{C}$ (ie, entire). Then if f is bounded, f is constant. Now this is an extraordinary result because you wouldn't think it would come from just being complex differentiable, and certainly not this easily.

Let z_1, z_2 be complex numbers. Pick $R > \max\{2|z_1|, 2|z_2|\}$.

Then $f(z_1) - f(z_2) = \frac{1}{2\pi i}\int_{|w|=R} \frac{f(w)}{w-z_1} - \frac{f(w)}{w-z_2} dw$. So $|f(z_1) - f(z_2)| = \frac{1}{2\pi}\left|\int_{|w|=R} \frac{f(w)(z_1-z_2)}{(w-z_1)(w-z_2)} dw\right|$

Now suppose M is a bound for f, then

$$|f(z_1) - f(z_2)| \leq \frac{1}{2\pi} 2\pi RM \frac{|z_1 - z_2|}{\left(\frac{R}{2}\right)^2}$$

Where this is $\frac{1}{2\pi}$ times the length of the path times the maximum of the function times another constant divided by a lower bound for the denominator. Now this goes to 0 as R goes to infinity, so if f is bounded, $f(z_1) = f(z_2)$ so f is constant.

Lecture 7:

Corrolary (Fundamental theorem of algebra): Every non-constant polynomial has a root in $\mathbb{C}$. By repeatedly factoring it out we see that every polynomial is a product of linear factors.

Proof: Suppose there were no roots, then its reciprocal is differentiable everywhere hence continuous everywhere. Therefore we have an entire function as the reciprocal. There is some ball such that outside it the reciprocal is less than 1 in magnitude, since polynomials go to infinity in size as x gets large once the most significant term dominates, and by the extreme value theorem the reciprocal of the polynomial achieves a maximum in that ball because it is continuous everywhere in that ball. But then this is a bounded non-constant entire function contradicting the previous theorem.

Theorem: The derivative of a holomorphic function is holomorphic. In fact this implies a holomorphic function is infinitely differentiable. But it gets even better than that. We will prove that if f is holomorphic on an open ball then f has a convergent power series representation on that ball.

We will show that $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ convergent on the open ball around a that f is holomorphic on of radius r, and that for 0<p<r we have $c_n = \frac{1}{2\pi i}\int_{\partial B_p(a)} \frac{f(w)}{(w-a)^{n+1}} dw$. Note that $c_0 = f(a)$ by the cauchy integral formula.

Proof: For this p, if z is such that $|z - a| < p$ then by the cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\int_{B_p(a)} \frac{f(w)}{w - z} dw$$

We also recall that $\frac{1}{w-z} = \frac{1}{(w-a)\left(1-\left(\frac{z-a}{w-a}\right)\right)}$.

As in last lecture we write $f(z) = \frac{1}{2\pi i}\int_{B_p(a)} f(w) \sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}} dw$

And again using uniform/dominated convergence to swap the sum and the integral, we get the result:

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_{\partial B_p(a)} \frac{f(w)}{(w-a)^{n+1}} dw\,(z-a)^n$$

As desired.

Remark: If f is holomorphic on a domain U then at any a in U there is an open ball around a in U such that f is holomorphic. So we have a convergent power series on this ball for sure, but not necessarily convergent everywhere on U.

Theorem (Morera's theorem): Let U be a domain and f be continuous on U. Suppose that f vanishes when you integrate it along any contour, then f is holomorphic on U.

Proof: We know from earlier that f has an antiderivative. But then this antiderivative is holomorphic on U, and thus so is f by the previous theorem.

Corollary: Let U be a domain and let f_n be a sequence of holomorphic functions on U and suppose they converge to some function f uniformly on U. Then f is holomorphic on U and its derivative is the limit of the derivatives of f_n.

Proof: Take U to be a disc of radius r about z in which f is holomorphic on the closed disc of radius r about z and prove it there, since we can just combine the results over all such discs since holomorphicity is a local property and any domain is a union of such discs (as an open disc is a union of closed discs, since an open ball of radius 1 is the union of one of radius 0.9, 0.99, 0.999, etc). If γ is a closed contour, then by Cauchy's theorem for a star shaped domain the integral of f_n vanishes on γ and since γ has finite arc length, uniform convergence implies that $\int_\gamma f_n \to \int_\gamma f$ (the integrals get within ε when f inevitably is in $\frac{\varepsilon}{length}$ everywhere by uniform convergence).

So the integral of f vanishes, so by Morera's theorem f is holomorphic.

By earlier, $f'(z) = \frac{1}{2\pi i}\int_{\partial B_r(z)} \frac{f(w)}{(w-z)^2} dw$

$$|f_n'(z) - f'(z)| = \frac{1}{2\pi}\left|\int_{\partial B_r(z)} \frac{f(w) - f_n(w)}{(w-z)^2} dw\right| \leq \frac{1}{2\pi} 2\pi r \frac{1}{r^2} \sup_{|w-z|=r} |f(w) - f_n(w)|$$

Which goes to 0 by uniform convergence.

Note that it is necessary that we have a domain. There is a theorem that we may or may not have seen in Analysis II but the statement is that any continuous function on [0,1] can be uniformly approximated by polynomials. This allows for non-holomorphic continuous functions on [0,1] such as functions with a spike or a fractal graph to be approximated by holomorphic functions that converge uniformly to it. The reason this is not a contradiction is because [0,1] is not a domain in $\mathbb{C}$ - remember, a domain has to be both path-connected and open, and [0,1] is closed – even if it were (0,1) which is open in $\mathbb{R}$ it is not open in $\mathbb{C}$ because it contains no open balls in the complex plane.

Corollary: If U is a domain and f is continuous and holomorphic on all but finitely many points, then f is holomorphic on U.

Proof: Since there are finitely many bad points, each point has a ball around it with no other bad points. Cauchy's theorem on this ball (since the ball is star shaped) says that any integral vanishes, then morera's theorem implies f is actually holomorphic on this ball.

Theorem (Cauchy's theorem for simply connected domains): If U in $\mathbb{C}$ is a simply connected domain and f is holomorphic on U then the integral of f vanishes on any closed contour on U.

We will develop some definitions to work towards a proof of this statement. The idea is that a loop is many loops in convex domains stitched together.

Let ϕ, ψ be closed paths in a domain U with domain of definition [a,b]. We say they're homotopic if there is a continuous map $\Phi: [a, b] \times [0,1] \to U$ such that if we set the second thing to 0 you get the path ϕ, if we set it to 1 we get ψ, and for each t from 0 to 1 we get a closed continuously differentiable path.

This works in simply connected domains: the loop can be continuously shrunk to a point which can be continuously unshrunk to the other loop, but it remains to show that if this can be done at all then this can be done in a fashion such that each loop is not only continuous but continuously differentiable.

Proposition: If I have two closed contours in a simply connected domain then I can deform one into the other continuous in a fashion such that each intermediate curve is a closed contour.

We use the fact that the domain is simply connected when we assume that any loop can be turned into any other in a continuous fashion – we can shrink one to a point then unshrink the point to the other loop.

Proof: Let $\gamma_0, \gamma_1: [0,1] \to D$ be your loops and (by simply-connected-ness) there is a continuous map H from $[0,1] \times [0,1] \to D$ such that $H(s, 0) = \gamma_0(s), H(s, 1) = \gamma_1(s)$.

Let K be the image of H, Then since $[0,1] \times [0,1]$ is compact, so is K. Since D is open, it follows that the least distance from K to the complement of D is bounded below by some positive δ (to see this, the minimum is attained by compactness + the extreme value theorem).

Now by uniform continuity, there is a t so small that $|H(y) - H(x)| < \frac{\delta}{4}$ (the 4 is arbitrary, the point is some fixed constant works) whenever $|x - y| < t$. Now take a partition $0 = t_0 < t_1 < \cdots < t_N = 1$ such that for each i from 0 to N-1, $t_{i+1} - t_i \leq t$.

Recall from a certain proof we did in differential equations that uniform continuity allows us to define a loop by linear interpolation between points which are close enough and then the new loop is guranteed to be within some threshold of the old loop.

Concretely, for each s, we make a loop P_s by connecting each $H(s, t_i)$ to $H(s, t_{i+1})$ with a straight line. By how we chose t, every line will be less than $\frac{\delta}{4}$ in length so we will never leave the open region.

Now what we can do is connect γ_0 to P_0 by simply moving each point to each other point in a straight line, and then we say that this new path is $H'[s, u]$ for some u in (0,1), say $u = \frac{1}{3}$.

Now define H'(s,t) staying in D by continuously deforming between our curves as follows:

$$\begin{cases} (1 - 3s)\gamma_0(t) + (3s)P_0(t) \; if \; 0 \leq s \leq \dfrac{1}{3} \\ P_{3s-1}(t) \; if \dfrac{1}{3} \leq s \leq \dfrac{2}{3} \\ (3 - 3s)P_1(t) + (3s - 2)\gamma_1(t) \; if \dfrac{2}{3} \leq s \leq 1 \end{cases}$$

Clearly each intermediate path is piecewise continuously differentiable since it is a linear combination of such paths.

Proposition: If two closed paths are homotopic, then the integral over these two paths are equal. Cauchy for almost simply connected domains follows by taking one of them to be a point. We will prove this next lecture.

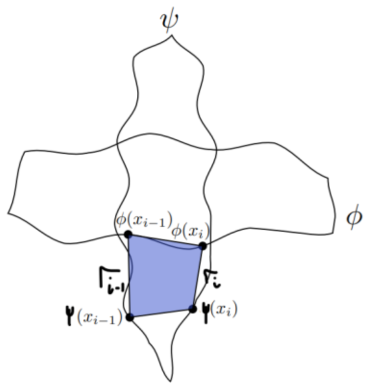
Lecture 8:

Proof of the proposition:

Let ϕ, ψ be closed contours. We say they are elementary deformations of one another if you can cut up your domain of definition [a,b] and convex open sets C_i in U such that if you take one part of your partition of $[a, b]$ it's image under both paths is entirely inside C_i.

By convexity, we let γ_i be a straight path between the two paths.

Let $\Gamma_i = \phi_i + \gamma_i - \psi_i - \gamma_{i-1}$ which anything we integrate along this is 0 by convex cauchy. Therefore, by summing over all i, we get that if two paths are elementary deformations of one another then the integral of anything along the two paths are the same.



This image shows what is going on.

Now we just need to show that there is a sequence of elementary definitions to get from any path to any other.

Let $H: [a, b] \times [0,1] \to U$ be our homotopy. Then we have a compact domain so its image must be compact by continuity (to see this, the image must have its limit points since the pre-image does and the function is continuous) and H is uniformly continuous. Also, the minimum distance from the image of H to the edge of U is positive (as in last lecture), say this is ε.

Now by uniform continuity of H there is some universal δ such that changing the path by δ and picking a sufficiently fine partition of [a,b] the two paths are elementary deformations using an open ball of radius ε. Ie, each little square in our partition of $[a, b] \times [0,1]$ has its image inside an ε ball, so any two paths within δ are elementary deformations. So we have our sequence of elementary deformations, as we wanted.

Now we talk about zeroes of holomorphic functions.

Let f be holomorphic on an open ball about a, so by taylor's theorem we know we can write f as a convergent power series.

If f is not 0 everywhere, we can write f as a power series $f(z) = (z - a)^m g(z)$ where g is not 0 at a. If m is not 0, then we say f has a 0 of order m at a.

Proposition: (Principle of isolated zeroes) If f is holomorphic on a ball about a, not 0 everywhere on the ball, then there exists some p such that on the ball of radius p about a not including a, f is not 0.

Proof: If f(a) is not 0 we are done by continuity of f. Otherwise, write $f(z) = (z-a)^m g(z)$ and apply the "trivial case" to g.

Theorem: (Identity theorem) Let U∈ℂ be a domain and let $f, g: U \to \mathbb{C}$ be holomorphic. Let S be the set of z in U such that $f(z) = g(z)$. Then if S has an accumulation point in U, then S=U.

Proof: Apply the previous proposition to f-g. Specifically, suppose w is an accumulation point of S, then f=g is 0 in some open ball around w. Now if z is in U, we can join z to w by a piecewise C^1 path ϒ.

Now consider $I = \{t \in [0,1]: all\ derivatives\ of\ f - g\ at\ \Upsilon(\mathrm{t}) = 0\}$. Then I=[0,1] because I is non-empty, closed (contains limit points), and open in [0,1] (by the principle of isolated zeroes).

Note that sets which are both open and closed in a connected set that are non-empty must be the entire set since otherwise we would have a splitting of the connected set into two disjoint non-empty sets (one the clopen set and one its complement which is clopen). Say U is a disjoint union of two open sets in ℂ. Now build a path segment from any point A in U to any point B in the other open part of U. Now for any point C on the path in A, it must have be that there is a disc around that point in A by openness, and thus an open interval around t must be a part of the path that is contained in A since the path is continuous so pre-images of open sets are open. But then at the point x in the path where $F(x) = Sup(F(t), t \in A)$, x cannot be in A otherwise it would be in A at a larger point, so x is in B, but then x is also in B in an interval around it, so we do not have anymore that that point is the least upper bound of the points in A anymore. Contradiction.

This means that on complex functions we cannot do that thing where we have a function that is 0 and then starts growing in a differentiable fashion like we could for real functions.

However, zeroes can accumulate at a point not in U, such as $\sin\left(\frac{1}{z}\right)$ for z not 0.

Lecture 9:

Notation: If U is a domain then $\overline{U}$ is U with its boundary, or the closure of U. We write ∂U for the boundary of U.

Corollary (Global maximum principle): Let U in ℂ be a bounded domain (ie is in some ball) and take a function which is holomorphic on U and continuous on ∂U. Then |f| achieves its maximum on ∂U.

Proof: $\overline{U}$ is closed and bounded and we know the extreme value theorem well, ie |f| achieves its maximum somewhere. If it achieves its maximum at some interior point, then it achieves its maximum in some open ball around a at exactly a, and therefore it is constant in this ball by the local maximum principle, so by the identity theorem f is constant on all of U. Therefore, if the theorem is not true, f is constant, in which case the theorem clearly is true.

Definition: Let $U \subseteq U' \subseteq C$ be domains and let $f: U \to \mathbb{C}$ be holomorphic. Then a holomorphic function $f: U' \to \mathbb{C}$ which is also holomorphic is called an analytic continuation of f. By the identity theorem we know this is unique if it exists, but it does not always exist.

Definition: If $f: U \to \mathbb{C}$ is holomorphic and has no analytic continuation to any larger open set then we say that ∂U is the natural boundary of f.

Example: Consider $f(z) = \sum_{n=0}^{\infty} z^n$ which has radius of convergence 1 and therefore defines a holomorphic function on the unit disc. But we know that this is the series expansion of $\frac{1}{1-z}$ so it has an analytic continuation $\frac{1}{1-z}$ defined on all of $\mathbb{C}\backslash\{1\}$.

This is an example of something more general: The radius of convergence of a power series is the distance to the nearest point in $\mathbb{C}$ where the analytic continuation does not exist, since the holomorphic functions on open balls and the convergent power series on these balls are the precisely the same functions.

Example: $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ cannot be extended beyond the open disc $B_1(0)$. This is essentially because the series blows up on the boundary of the disc at every point whos argument is of the form $\frac{c\pi}{2^k}$ so you can never dodge these points when trying to make an analytic continuation.

Recall that if U is a domain and f is continuous on U and holomorphic everywhere except for on a finite set then f is holomorphic on U. We will investigate what happens if f is not continuous at some point S.

Proposition (Removal of singularities): Let U be a domain, $z_0 \in U$, and let f be holomorphic on $U\backslash z_0$ but bounded in some neighbourhood of z_0, then f can be analytically continued to z_0.

An example of this is extending $\frac{x}{x}$ to be 1 at the point x=0.

Proof: Consider the function that is 0 at z_0 and $(z - z_0)^2 f(z)$ otherwise, which we know is continuous at z_0 and holomorphic around it since f is bounded. Recall that a function continuous and holomorphic except at a single point is actually holomorphic. Therefore $(z - z_0)^2 f(z)$ is holomorphic at z_0 and because f is bounded this has a taylor series expansion with first 2 terms necessarily equal to 0, so we can use this to get a taylor expansion for f, hence f is holomorphic.

Proposition: Let U be a domain, $z_0 \in U$, and $f: U\backslash\{z_0\} \to \mathbb{C}$ be holomorphic. Suppose that |f| goes to infinity as $z \to z_0$. Then there is a holomorphic function $g: U \to \mathbb{C}$ and a $k \in \mathbb{N}$ such that $f(z) = \frac{g(z)}{(z-z_0)^k}$ and both g and k are unique.

Proof: Pick δ>0 such that $|f(z)| \geq 1$ on $B_\delta(z_0)\backslash z_0$. Then $h(z) := \frac{1}{f(z)} \to 0$ as $z \to z_0$ and by the previous proposition $h(z)$ is holomorphic on $B_\delta(z_0)$. But then $h(z) = (z - z_0)^k l(z)$ where l is holomorphic on this ball and not 0 at z_0 (by continuity it is also not 0 sufficiently close to z_0), so clearly k is unique, now set $g = \frac{1}{l}$ and the result follows, and g must also be unique. Also g is holomorphic because it is the reciprocal of a non-zero holomorphic function on some open disc.

The converse is clearly true because g is bounded near z_0 but $(z - z_0)^k \to 0$.

Definition: Let U be a domain, $z_0 \in U$, and $f: U\backslash\{z_0\} \to \mathbb{C}$ be holomorphic. We say that f has an isolated singularity at z_0. We classify this into one of three types:

1. Removable singularity (If f is bounded near the point)
2. Pole (If f goes to infinity near the point), we say it is of order k where k is defined as in the previous proposition
3. Essential (Otherwise), exactly when $|f|$ has no limit near this point.

An example of an essential singularity is $e^{\frac{1}{z}}$ at z=0.

Lecture 10:

Essential singularities are bad spectacularly.

Proposition (Casorati-weierstrass theorem): Near an essential singularity, for any complex number z there exists a sequence of points approaching the essential singularity such that f approaches z.

In fact there is a theorem (picard's theorem) that says actually f takes all but at most 1 value on every neighbourhood, but we don't need this stronger theorem and its proof is beyond the course, so we will only prove the casorati-weierstrass theorem. There is a deeper idea that somehow $\mathbb{C}$ and $\mathbb{C}$ minus 1 point and $\mathbb{C}$ minus more than 1 point behave completely differently.

Proof: The idea is that if this were not the case then $\frac{1}{f-z}$ would be bounded near the singularity and therefore holomorphic there so f would have either a removable singularity or a pole at that point, which is a contradiction.

As an example, if we take z=2 and the essential singularity at 0 of $e^{\frac{1}{z}}$, then the sequence $\frac{1}{2ni\pi+\ln(2)}$ works.

Another way to see this is that any little open ball is sent by $\frac{1}{z}$ to the compliment of some closed ball which contains some 2π width strip that is sent to every value near $z = 0$.

Example: $\frac{\sin(z)}{z}$ has a removable singularity at 0 because we can just make it 1 and then it has a power series $1 - \frac{z^2}{6} + \frac{z^4}{120} - \cdots$.

Example: $\frac{1}{z^3}$ has an order 3 pole at the origin.

Note that if we add the point at infinity, then near a pole if we add the point at infinity we have a perfectly well defined function. We say f is meromorphic on a domain U if on all but a set of isolated points it is holomorphic and at these isolated points it has at worst poles and removable singularities. A classic example is a ratio of polynomials.

We will now talk about functions which are holomorphic on a punctured disc (meaning a ball not including its middle).

Theorem (Laurent series): Let A be an annulus (meaning a disc minus a smaller disc, it could be the case that the larger disc is infinite and/or the smaller disc is just a point) and let $f: A \to \mathbb{C}$ be holomorphic, then f has a unique convergent series expansion of the form $\sum_{n=-\infty}^{\infty} c_n(z-a)^n$. Furthermore this converges uniformly for closed bounded sets inside A.

Furthermore, for any radius p in the annulus, $c_n = \frac{1}{2\pi i}\int_{\partial B_p(a)} \frac{f(w)}{(w-a)^{n+1}} dw$. However, we will not get an easier formula involving the derivative like there is for Taylor's theorem.

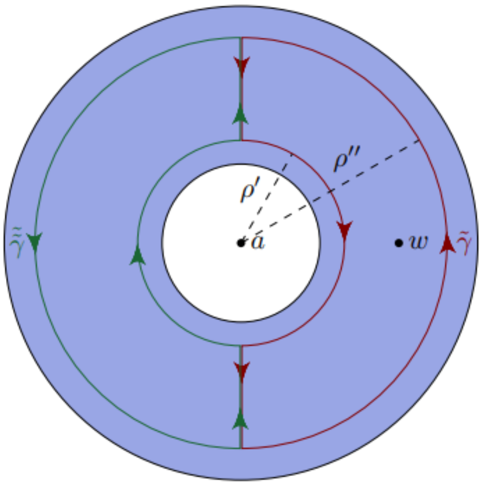
Once we prove this we immediately see that the following hold (for laurent series on punctured discs where we have a singularity in the middle)

i) a is removable if and only if all the negative coefficients are 0
ii) a is a pole if and only if finitely many of the negative coefficients are non-zero, and the order is k if and only if k is the lowest non-zero negative coefficient

iii) a is an essential singularity if and only if infinitely many of the negative coefficients are non-zero

Proof: Fix some closed bounded annulus $p_1 \leq r \leq p_2$ inside the main annulus.

By considering the red and green curves in this image and the fact that they must both go to 0 by Cauchy's theorem, we know that the integral of some holomorphic function f along one of these circles is independent of the circle's radius.



So let z be any point with radius r. Then by Cauchy's integral formula (whoch holds for non-balls because of convex deformation discussion), we know that if we take $\frac{f(w)}{w-z}$ as a function of w, then this will integrate to 0 on the path that does not cnclose z and f(z) on the other path. Therefore we get that

$$f(z) = \frac{1}{2\pi i}\left[\int_{\partial B_{p_2}(a)} \frac{f(w)}{w-z} dw - \int_{\partial B_{p_1}(a)} \frac{f(w)}{w-z} dw\right]$$

For the first integral we write

$$\frac{1}{w-z} = \frac{1}{w-a}\left(\frac{1}{1-\frac{z-a}{w-a}}\right) = \sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}}$$

Which is uniformly convergent since $|z-a| < |w-a|$.

Similarly,

$$\frac{-1}{w-z} = \frac{1}{z-a}\left(\frac{1}{1-\frac{w-a}{z-a}}\right) = \sum_{m=1}^{\infty} \frac{(w-a)^{m-1}}{(z-a)^m}$$

And again this converges uniformly.

We can now swap the sum and integral to get

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty} \int_{\partial B_{p_2}(a)} \frac{f(w)}{(w-a)^{n+1}} dw\,(z-a)^n + \frac{1}{2\pi i}\sum_{m=1}^{\infty} \int_{\partial B_{p_1}(a)} \frac{f(w)}{(w-a)^{-m+1}} dw\,(z-a)^{-m}$$

Subsituting n with m in the second case gives the result (it is ok that the radii are different because the integral of any function which is holomorphic everywhere on the annulus on any circle centered about a is the same by previous discussion).

To prove that the coefficients are uniquely determined by f, suppose that for some k, we have c_k given by the integral formula and another valid coefficient b_k. Then we have

$$2\pi i c_k = \int_{\partial B_p(a)} \frac{f(w)}{(w-a)^{k+1}} dw$$

We can substitute $f(w) = \sum_{n=-\infty}^{\infty} b_n (w-a)^n$ to get

$$2\pi i c_k = \int_{\partial B_p(a)} \sum_{n=-\infty}^{\infty} b_n (w-a)^{n-k-1} dw$$

Again we have that if $\sum_{n=-\infty}^{\infty} b_n (w-a)^n$ converges then it converges uniformly (since the positive and negative terms converge uniformly by previous discussion), so we can write

$$2\pi i c_k = \sum_{n=-\infty}^{\infty} \int_{\partial B_p(a)} b_n (w-a)^{n-k-1} dw$$

And we know that the only term that survives is the n=k term, again by previous discussion, and here the integral comes out to exactly $2\pi i b_k$ so we have uniqueness.

Lecture 11:

We say a laurent series converges when the positive powers and the negative powers converge separately. But in order to rearrange terms like this, we need absolute convergence, ie we need to be inside the radius of convergence of the power series convergence, and outside $\frac{1}{R}$ where R is the radius of convergence of the negative part. We have the negative part is $\sum_{n=1}^{\infty} c_{-n}(z-a)^{-n}$ so we need to have that $|(z-a)^{-1}| < R$ so $|z-a| > \frac{1}{R}$. If these constraints actually intersect we have a series that converges on an annulus. But on the other hand, if a function is holomorphic on an annulus it has a convergent laurent series expansion.

Definition: The principal part of a laurent series is the part with negative powers. This is always equal to 0 at $z = \infty$.

Example: $\csc\left(\frac{1}{z}\right)$ has singularities at $\frac{1}{k\pi}$ so it has a non-isolated singularity at 0 so it has no laurent series on any punctured disc around the origin.

Example: Consider $\csc(z) = \frac{1}{\sin(z)}$ which is holomorphic except at multiples of π. This therefore has a laurent series on a punctured neighbourhood of 0. We know that $\sin(z) = z\left(1 - \frac{z^2}{6} + O(z^4)\right)$.

Therefore $\csc(z) = \frac{1}{z}\left(1 - \frac{z^2}{6} + O(z^4)\right)^{-1} = \frac{1}{z}\left(1 + \frac{z^2}{6} + O(z^4)\right)$ by just trying to do some kind of binomial series expansion of $\left(1 - \frac{z^2}{6} + O(z^4)\right)^{-1}$.

Example: $\sum_{n=-\infty}^{\infty}\frac{1}{(z-n)^2}=\frac{\pi^2}{\sin^2(\pi z)}$ for z in ℂ\ℤ.

Proof: Let $f(z)=\sum_{n=-\infty}^{\infty}\frac{1}{(z-n)^2}$. If w is in ℂ\ℤ there is some r>0 such that $|w-n|>2r$ for all n. If z is in the radius r disc around w then $|z|\leq|w|+|z-w|\leq|w|+r$ and $|z-n|\geq|n|-|z|$ because by the triangle inequality $|n|\leq|n-z|+|z|$. We therefore get $|z-n|\geq|n|-|w|-r\geq n-|w|-r$ and we also get that $|z-n|\geq r$. We therefore conclude that $\frac{1}{|z-n|^2}\leq\min\left\{\frac{1}{r^2},\frac{1}{(n-|w|-r)^2}\right\}$. But now it is easy to see that $\sum\frac{1}{|z-n|^2}$ is bounded baove by something absolutely convergent, and it therefore follows that the series $\sum\frac{1}{(z-n)^2}$ has a minimum rate of convergence and converges uniformly on the ball. This is since uniform convergence happens if there is a minimum rate of convergence that everything is bounded below by, an intuitive fact that is good enough to understand why what we are saying is true, but that we will make more precise in other courses when we talk about the Weierstrass M-test.

We know though that uniform limits of holomorphic functions are holomorphic, so f is holomorphic on the ball. Since w was arbitrary we know f is holomorphic on ℂ\ℤ (f is not uniformly convergent there but it is holomorphic). Now note that $f(z+1)=f(z)$ everywhere and that f has a pole of order 2 at every integer. We also know that $\frac{1}{\sin^2(\pi z)}$ has a double pole at each integer. We know that near the origin f is $\frac{1}{z^2}+Holomorphic\ stuff$, so near each k∈ℤ, the principal part of the laurent series is $\frac{1}{(z-k)^2}$. But we know that singularities with the same principal part happen in the function $g(z)=\left(\frac{\pi}{\sin(\pi z)}\right)^2$. We know now that $h(z)=f(z)-g(z)$ has only removable singularities at the integers, so it is an entire function. We first want to show that h is bounded (since it would therefore be constant). By periodicity it suffices to show that if $-\frac{1}{2}\leq Re(z)\leq\frac{1}{2}$ then h is bounded. Note that by continuity, it is bounded on any rectangle of width 1, so we just need to show that $h(x+iy)\to 0$ as $y\to\pm\infty$. We note that $|g(z)|=\frac{4\pi^2}{|e^{\pi y}-e^{-\pi y}|}\to 0$, and since we know that the real part x has $-\frac{1}{2}\leq Re(z)\leq\frac{1}{2}$ we can conclude that $|f(z)|\leq\sum_{n=-\infty}^{\infty}\frac{1}{|x+iy-n|^2}\leq\frac{1}{y^2}+2\sum_{n=1}^{\infty}\frac{1}{\left(n-\frac{1}{2}\right)^2+y^2}\to 0$. Therefore h is bounded on the strip, hence constant, and because of our arguments this constant is in fact equal to 0, so the result follows.

This implies with some careful work the famous result $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$.

We will now talk about winding number, ie how to formalize the number of times a loop encloses the origin.

Let Y be a path from [a,b] to ℂ not including the origin.

Lemma: There are continuous functions r and θ such that $\gamma(t)=r(t)e^{i\theta(t)}$.

Proof: Set $r(t)=|\gamma(t)|$ which is clearly continuous and is never 0 by the extreme value theorem and the fact that the path encloses the origin (we will need the non zero bit in the next sentence).

Now we replace γ with $\frac{\gamma}{r}$ so our path takes values in the unit circle.

If the path is contained in $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$ we can set $\theta(t)=Arg\big(\gamma(t)\big)$. In fact we could do this on any slit plane using any branch of the argument. Any two different θ's will differ by at worst a multiple of 2π.

Since γ is continuous and so uniformly continuous on [a,b] there exists some δ such that if $|s-t|<\delta$ then $|\gamma(s)-\gamma(t)|<2$ so they are not opposite on the unit circle. Now split the interval into a bunch of δ intervals, such that for each of these intervals, γ has an image in a slit plane.

We now have made it so that on any 2 consecuative intervals of the partition, the argument agrees on some part of them, and so now it is clear how this idea is formalized.

Lecture 12:

We now know if we have a closed path that misses a point w how to formalize the idea of how many times your path encloses w. We define the winding number of a closed path Y about w which we write as $I(\gamma, w)$ as the number of times we enclose w anticlockwise, ie the argument difference divided by 2π.

If a path is piecewise C^1 and closed and does not pass w then an alternative definition of this is given by $\frac{1}{2\pi i}\int_\gamma \frac{1}{z-w}dz$, because of how the imaginary part of the logarithm changes in the same way as the argument.

We will now make an observation which we will not fully formalize because formalizing it is not really necessary to understand why it is true. If we take a path and consider a region it encloses, the winding number will be constant inside each such region. There is exactly one unbounded region (ie the region not enclosed by the path) where the winding number is 0.

We will, however, see a formalization of this for piecewise C^1 paths.

The integral form of the winding number is continuous in w whenever w does not cross the path, but it is integer valued, so it must be constant.

Also, the same is true for the unbounded section, except this time the integer must be 0 because $\frac{1}{z-w}$ can get as small as we like by making w sufficiently large, or equivalently because we are integrating a well defined holomorphic function on a simply connected domain on a closed loop.

Lets go back to holomorphic functions on punctured discs with an isolated singularity at a and therefore a local Laurent series. In $\sum_{n=-\infty}^{\infty} c_n(z-a)^n$, when we integrate this in some disc around a, every term except the $n=-1$ term will vanish. We define c_{-1} to be the residue of f at a. We get the following result:

Theorem (Cauchy's Residue Theorem): Let U be a simply connected domain and let f be holomorphic everywhere except on a finite set with points $z_1, z_2, \dots, z_k$ in U (where it may have isolated singularities). Suppose we have a closed path γ such that none of these points lie on the path. Then

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_{i=1}^{k} I(\gamma, z_i) res_f(z_i)$$

Proof: There is a laurent series with principal part g_j in a punctured disc about each z_j. If we take a new function $f-g_1-g_2-\cdots-g_k$ then this must be holomorphic everywhere on U (as it just has removable singularities).

By the other cauchy theorem for simply connected domains the integral of $f-g_1-g_2-\cdots-g_k$ is 0 on any contour. Therefore $\int_\gamma f(z)dz = \sum_{i=1}^{k}\int_\gamma g_i(z)dz$. Because we can swap the sum and integral by uniform convergence of g_i on compact subsets of U, we notice that only the integral of the -1 power term of the laurent series will survive. Therefore, the result follows.

Note that we get the cauchy integral formula by applying Cauchy's residue theorem to $\frac{f(z)}{z-w}$.

We say a contour bounds a domain D in U if the winding number is 1 for w in D and 0 otherwise.

The residue theorem says that integrating over a loop counts the residue of all the singularities the loop encloses.

Lemma: Let U be a domain and let f be meromorphic with a pole at a. Then

i) If this pole has order 1 then $\text{res}_f(a) = \lim_{z \to a}(z-a)f(z)$

ii) If $f(z) = \frac{g(z)}{h(z)}$ where g and h are holomorphic at a and $g(a) \neq 0$ and h has a zero of order 1 at a then $\text{res}_f(a) = \frac{g(a)}{h'(a)}$.

iii) If $f(z) = \frac{g(z)}{(z-a)^k}$ with g holomorphic near a then $\text{res}_f(a) = \frac{g^{(k-1)}(a)}{(k-1)!}$

Lecture 13:

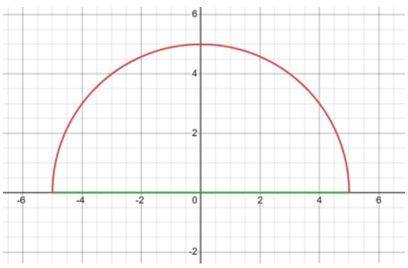
Proof:

To prove (i), just consider multiplying the laurent series expansion by z-a and taking a limit by continuity.

To prove (ii), we note that $\text{res}_f(a) = \lim_{z \to a}(z-a)\frac{g(z)}{h(z)}$ by (i) $= \lim_{z \to a}(z-a)\frac{g(a)}{h(z)-h(a)}$ because g is holomorphic and h(a)=0 so $\text{res}_f(a) = g(a)\lim_{z \to a}\frac{z-a}{h(z)-h(a)} = \frac{g(a)}{h'(a)}$.
To prove (iii), note that we just want the coefficient of $(z-a)^{k-1}$ in the taylor series for g, so the result follows.

Example: We will calculate $\int_0^\infty \frac{1}{1+x^4}dx$.

The function $f(z) = \frac{1}{1+z^4}$ has poles at $\frac{1}{\sqrt{2}}(\pm 1 \pm i)$. Consider a contour like this traversed anticlockwise:



We want to take a limit as we make this a larger semicircle. This encloses 2 poles and its winding number about each is +1. But note that the length of the semicircular bit is $O(x)$ and we are integrating an $O(x^{-4})$ thing so it goes to 0. But by Cauchy's residue theorem, our 2 poles have some residue which we will calculate, and the result will be $2\pi i * (The\ sum\ of\ these\ 2\ residues)$, but we have to divide by 2 since we are integrating from 0 to infinity, and the function is symmetric.

By part (ii) of our previous lemma, the residue of f at a pole p_i is $\frac{1}{4p_i^3} = -\frac{1}{4}p_i$ since $p_i^4 = -1$. So now we have that $2\int_0^\infty \frac{1}{1+x^4}dx = 2\pi i\left(\frac{1}{4\left(\frac{1}{\sqrt{2}}(1+i)\right)^3} + \frac{1}{4\left(\frac{1}{\sqrt{2}}(-1+i)\right)^3}\right)$. This looks horrible but we know that the poles are just at $e^{\frac{n\pi}{4}}$. We can simplify and check that it gives the answer $\frac{\pi}{2\sqrt{2}}$.

Example:

Lets find $\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x+x^2}dx$.

Note that high up on the imaginary axis, $\cos(x)$ grows exponentially. Therefore the same semicircular contour won't work. Let $f(z) = \frac{e^{iz}}{1+z+z^2}$ because if we integrate this the real part will be exactly what we want.

This will have poles at $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. We will consider the same semicircle, and the only pole we enclose will be the one at $\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. Lets consider the integral along the semicircle if its radius is R: The absolute value of that will be $\left|\int_0^\pi f(Re^{i\theta})iRe^{i\theta}d\theta\right| = \int_0^\pi \frac{Re^{-R\sin(\theta)}}{|R^2e^{2i\theta}+Re^{i\theta}+1|}d\theta$. But now note that this is $O\left(\frac{1}{R}\right)$ which goes to 0 so we just need to find $2\pi i * The\ residue\ at\ the\ pole$ and take the real part.

Now applying part (ii) of the lemma again we get $\frac{e^{iz}}{2z+1}$ is the residue and then we get the result

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x+x^2}dx = \frac{2\pi}{\sqrt{3}}\cos\left(\frac{1}{2}\right)e^{-\frac{\sqrt{3}}{2}}$$

Lemma: Let f have an order 1pole at a and be holomorphic on the open ball of radius r around a. For $0 < \varepsilon < r, \gamma_\varepsilon(t) = a + \varepsilon e^{it}$ defined on [A,B]. Then $\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} f(z)dz = (B-A)i * res_f(a)$

Proof: As f has an order 1 pole we can write $f = \frac{c}{z-a} + g(z)$ with g holomorphic. Then g is continuous and locally bounded so $\left|\int_{\gamma_\varepsilon} g(z)dz\right| \to 0$ as $\varepsilon \to 0$. But we know c is the residue of f, and that if we integrate $\frac{c}{z-a}$ on this arc we get $ic(B-A)$. So the result follows.

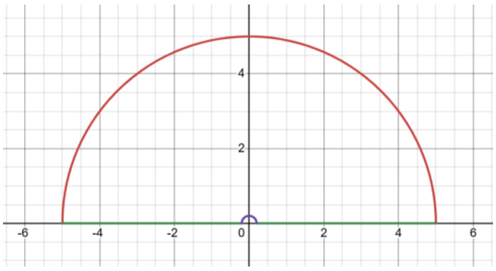
Lemma: If f is holomorphic on $\{|z| \geq r\}$ and $zf(z)$ is bounded for large $|z|$, then for all A>0, we have that $\int_{\gamma_R} f(z)e^{iAz}dz \to 0$ as $R \to \infty$ and γ_R is the semicircular arc of radius R.

Proof:

For large |z| there is some C such that $|f(z)| \leq \frac{C}{|z|}$. If $z = Re^{it}$ then $\left|e^{iAz}\right| = e^{Re(iARe^{it})} = e^{-RA*\sin(t)}$. Therefore if $0 \leq t \leq \frac{\pi}{2}$ then $\left|e^{iAz}\right| \leq e^{-RA*\frac{2t}{\pi}}$. If we can show that the integral along this quarter circle goes to 0 then the same argument will show that the integral along the whole semicircle goes to 0 by combining the 2 parts. Now the integral of the quarter arc is $\int_0^{\frac{\pi}{2}} e^{iRAe^{it}}f(Re^{it})Re^{it}\,dt$. But now this is bounded in absolute value by $\int_0^{\frac{\pi}{2}} e^{-RA*\frac{2t}{\pi}}\frac{C}{R}R\ dt = C\int_0^{\frac{\pi}{2}} e^{-RA*\frac{2t}{\pi}}dt\ = \frac{C}{2R}(1-e^{AR}) \to 0$.

Example: $\int_0^\infty \frac{\sin(x)}{x}dx$

This has a removable singularity at 0, but it still becomes exponentially large on any semicirular contour. Let $f(z) = \frac{e^{iz}}{z}$, but then this has a pole at 0 with residue 1, so we cannot use a contour that goes through 0. Consider a contour like this traversed anticlockwise:



Now the integral over this contour is 0 because it encloses no poles.

Now consider the integral over the small semicircle in the limit. By 2 lemmas ago, it is i*res*(B-A) where here because we are going anticlockwise, B-A=-π. We therefore get $-i\pi$. And by the lemma 1 lemma ago, the integral over the large semicircle goes to 0. So overall in the limit, we have

$$\int_{-\infty}^{\infty} f(z)dz = i\pi$$

But we will get minus the same result if we integrate $f(-z)$, and sin is $\frac{f(z)-f(-z)}{2i}$. Therefore

$$\int_{-\infty}^{\infty} \frac{\sin(z)}{z} dz = \pi$$

So by symmetry,

$$\int_{0}^{\infty} \frac{\sin(z)}{z} dz = \frac{\pi}{2}$$

Example: $\int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh(x)} dx, -1 < a < 1$

This has poles whenever $z = \left(n + \frac{1}{2}\right) i\pi$ for some integer n. But we have the useful property that $\cosh(x + i\pi) = -\cosh(x)$.

Now consider a box contour where we traverse the boundary of a box anticlockwise where the box goes from -R to R on the real axis and 0 to π on the imaginary axis.

Along the right hand vertical segment, we have $\int_0^{\pi} \frac{e^{a(R+iy)}}{\cosh(R+iy)} dx \leq \int_0^{\pi} \left|\frac{2e^{aR}}{e^{R}-e^{-R}}\right| dx \to 0$ because a is between -1 and 1. But now note that if I is the integral on the bottom contour, then by the periodicity property of cosh, the integral on the top contour is $e^{a\pi i}I$. We can calculate the residue at the pole we enclose is $-ie^{\frac{a\pi i}{2}}$, again using the derivative formula. Therefore $\left(1 + e^{a\pi i}\right)I$ is 2πi times the residue. We can therefore calculate I and use trig identites to get $\pi \sec\left(\frac{\pi a}{2}\right)$.

Lecture 14:

Example: Lets find $\int_0^{\infty} \frac{\log(x)}{1+x^2} dx$.

In this case we will define a branch of the logarithm by cutting along the negative imagniary axis. Then we will consider a contour like the one we did for the $\frac{\sin(x)}{x}$ example. As usual we will find the limit as the semicircles get small and large On the large semicircle, the integral is on the order of $\frac{Rlog(R)}{R^2} \to 0$. On the small semicircle the integral is of order $\varepsilon \log(\varepsilon) \to 0$. But the only singularity we enclose is i, so the total integral on the real axis is $2\pi i * Res_f(i)$.

This means $2\int_0^\infty \frac{\log(x)}{1+x^2}dx + \int_{-\infty}^0 \frac{\mathrm{i}\pi}{1+x^2}dx = 2\pi i * Res_f(i)$. But the residue at i is $\frac{\pi}{4}$ by the lemma from lecture 12, and $\int_{-\infty}^0 \frac{\mathrm{i}\pi}{1+x^2}dx = \frac{i\pi^2}{2}$. Therefore we deduce that in fact $\int_0^\infty \frac{\log(x)}{1+x^2}dx = 0$.

We could also do a keyhole contour where we avoid a slit of the plane by a small amount and take a limit.

Definition: The zeroes and poles of a meromorphic function always have an order, as we have discussed before. It is exactly the smallest non-zero coefficient in the laurent/taylor series.

Lemma: Suppose f is meromorphic on a domain U with a zero of orker k or pole of order -k at a, then $\frac{f'(z)}{f(z)}$ has a pole of order 1 at a with residue k, just by the power rule for differentiation. Specifically, if $f(z) = (z-a)^k g(z)$ with g holomorphic and $g(a) \neq 0$ then $f'(z) = k(z-a)^{k-1} + (z-a)^k g'(z)$ so therefore $\frac{f'(z)}{f(z)} = \frac{k}{z-a} + \frac{g'(z)}{g(z)}$, and the second term is holomorphic near a.

We usually call $\frac{f'(z)}{f(z)}$ the logarithmic derivative of f (since it is the derivative of log f on any simply connected domain not containing 0).

Theorem (Argument principle): Let U be a simply connected domain And let f be a meromorphic function on U with finitely many zeroes and poles. Let Y be a closed contour that does not cross any zeroes or poles. Then $\int_\gamma \frac{f'(z)}{f(z)}dz$ is the sum of the orders of the zeroes minus the sum of the orders of the poles, each weighted by winding number.

Proof: This follows from Cauchy's residue theorem and from the previous lemma.

Remark: If we take $\Gamma = f \circ \gamma$ this is a closed path in $\mathbb{C}\backslash\{0\}$. Then$\int_\gamma \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{\mathrm{Im}_f(\gamma)} \frac{1}{f}df$ which is the number of times Γ encloses the origin.

Theorem (Rouche's theorem): Let U in $\mathbb{C}$ be a domain, let Y in U be a closed curve which bounds a domain D in U. Let f and g be holomorphic on U and suppose $|f(z)| > |g(z)|$ everywhere on Y. Then f and f+g have the same number of zeroes in D when counted with multiplicity.

Proof: If $|f(z)| > |g(z)|$ everywhere on Y then in particular neither f nor $f+g$ can be 0 on Y as f cannot be 0 or -g since the inequalities are strict. Therefore the integrals we will work with are well defined. Look at $h(z) = \frac{f(z)+g(z)}{f(z)} = 1 + \frac{g(z)}{f(z)}$. Note that for all z in Y, $h(z) \in B_1(1)$. Therefore the winding number of $h \circ \Upsilon$ about 0 is 0. So the argument principle implies that the number of zeroes and poles of h in D are the same counted with multiplicity. But the zeroes of h are the zeroes of f+g and the poles of h are the poles of f, and if f and f+g share a 0 then if they have the same multiplicity we will have no contribution, and if they have differnet multiplicities we will pick up their difference.

Lecture 15:

Example:

$z^4 + 6z + 3$ has exactly 3 roots counted with multiplicity in the annulus $1 < |z| < 2$.

The proof of this is as follows:

On $|z| = 2$, $|z^4| = 16 > 6|z| + 3 \geq |6z + 3|$. Therefore our polynomial has the same number of roots in $|z| < 2$ as x^4 which we know has 4.

However, on $|z| = 1$ by considering $6|z| \geq |z^4 + 3|$ and there is the same number of roots in our polynomial as in the polynomial 6z which only has 1 root.

So the result follows. Note that there are no roots on the boundary since the inequalities were strict.

Example:

Let P(x) be a polynomial with degree n with leading coefficient 1 and integer coefficients. Suppose that the constant coefficient is not 0 and that the n-1 coefficient is greater than the sum of the absolute value of all the other coefficients. Then P has no integer roots, and P cannot be factored into 2 integer polynomials.

Proof:

Consider $f(z) = a_{n-1}z^{n-1}$ and $g(z) = P(z) - f(z)$. Then we know $f > g$ on the unit circle by hypotheses. But then f and p have the same number of roots inside here. Therefore P has exactly 1 root outside the unit circle.

But now suppose for a contradiction that $P(x) = Q(x)R(x)$ with Q and R having integer coefficients. Suppose Q has the special root with size at least 1. Note that neither Q or R have vanishing constant term and they have integer coefficients. For each polynomial, the product of all its roots is an integer, but one has only roots strictly between 0 and 1 in size, so this is a contradiction.

Example:

$$\int_{|z|=2} \frac{1}{z^5 + 12z - 5} dz = 0$$

Proof:

On the contour, $|z^5| = 32 > 29 = 12|z| + |5| \geq |12z - 5|$. So all the poles are enclosed by the contour, by Rouche's theorem and the same argument as what we did with the degree 4 polynomial.

Note, however, that if we were integrating over a really large circle, we would be integrating something of $O(R^{-5})$ over a contour of size $O(R)$, so it certainly tends to 0. But it encloses the same poles, so the result follows by Cauchy's residue theorem.

Defintion: Let $a \in \mathbb{C}$ and let f be non-constant on $B_r(a)$. Then the local degree of f, which we write as $\deg_{z=a} f$ at a is the order of the 0 of $f(z) - f(a)$ at a.

Lemma:

Let f be holomorphic and non-constant near a. If $\gamma(t) = a + re^{it}$ with $0 \leq t \leq 2\pi$ and r sufficiently small, then the local degree of f at a is the winding number of $f \circ \gamma$ about f(a).

Proof:

By the principle of isolated zeroes, $f(z) - f(a)$ does not vanish on some punctured neighbourhood of a. So we can take r smaller than this threshold. Then the winding number is actually well defined. Now note that the winding number about 0 of $\left(f(z) - f(a)\right) \circ \gamma$ is the same as the winding number we want, and the result follows by the argument principle.

Theorem: (Local mapping degree) Take a holomorphic non-constant function on a ball as in the previous theorem and with local degree at a equal to $k > 0$. Then for each p sufficiently small, there is an ε such that if $0 < |w - f(a)| < \varepsilon, f(z) = w$ has k roots of order 1 inside the p-disc about a.

Intuitively this is true by approximating by a polynomial z^k.

Proof:

Again pick p small enough such that by the principle of isolated zeroes, $f(z) - f(a)$ and $f'(z)$ are not 0 in a punctured disc of radius p about a.

Let Y be the circle of radius p about a. Ie, $a + pe^{2i\pi t}$ for $0 \leq t \leq 1$. Then $f(a)$ is not in the image of $f \circ \gamma$. But this image is a closed set so there is some ball about a of radius ε that is not in the image. If w is in $B_\varepsilon\left(f(a)\right)$ we will count solutions to $f(z) = w$ inside $B_p(a)$.

By the argument principle, the number of zeroes is the winding number of $f \circ \gamma$ about w, which is the same as its winding number about f(a), since f(a) and w have the same winding number as we can get from a to w continuously without crossing through the path, in fact a straight line would work by our previous argument. But by the previous lemma, this is just k. So the result follows.

Theorem (Open mapping theorem): Let U be a domain and let $f: U \rightarrow \mathbb{C}$ be holomorphic and non-constant. Then the image of U is an open set.

Proof: For each a in U, since U is open, we can find such a p-disc as in the above theorem, and then what happens is that there is an ε-disc about f(a) that is in the image, again by the above theorem. So the image is open.

Furthermore, the image of f under any open V in U is open.

Corollary: Let U be a simply connected domain not equal to ℂ. Then there is a holomorphic injection frun U to the unit disc. Note that this is false if U=ℂ by Liouville's theoorem.

Proof:

Let q be a point in ℂ not in U, then consider a branch of the logarithm in U by $\log(z - q)$, possible because U is simply-connected. So we can solve the equation $z - q = h^2$ for h holomorphic on U, by just taking $e^{\frac{1}{2}\log(z-q)}$. Now lets pick some point y in $h(u)$, then there is an open neighbourhood of this in $h(u)$ by the open mapping theorem.

Note that $y \cap B_\varepsilon(-y)$ is empty if ε is sufficiently small. Essentially, h is one branch of the square root so if $B_\varepsilon(y)$ is in h(u), $B_\varepsilon(-y)$ cannot be in h(u) at all since if something and –(that something) were in h(u), they would be the square root of the same thing, so we would have a multivalued function.

Now consider $f(z) = \frac{\varepsilon}{2(h(z)+y)}$, then this is bounded by 1, and it is invertible so it is an injection.

Lecture 16:

As the last theorem in this course, we will try to prove the riemann mapping theorem, which says that every simply connected domain that is not equal to $\mathbb{C}$ is conformally equivalent to the open ball $B_1(0)$. Note that the not equal condition is necessary because otherwise we would have a holomorphic non-constant function from $\mathbb{C} \to B_1(0)$ which we know is not possible.

Definition: We say a sequence of holomorphic functions converges locally uniformly to some f if at every point they converge uniformly on some ball around that point.

Example: z^n as n gets large converges not uniformly but locally uniformly to 0 on $B_1(0)$.

Proposition: If U is a domain and $f_n \to f$ locally uniformly on U where the f_n are holomorphic, then f is holomorphic and $f_n' \to f'$ locally uniformly.

Proof:

Recall that we proved a similar statement in lecture 7. Recall also the integral forms of the terms in the tatlor series.

Fix z and let r be constant such that f converges uniformly on $\partial B_r(z)$.

So if $|z-a| < \frac{r}{2}$ then

$$|f_n'(z) - f'(z)| = \frac{1}{2\pi}\left|\int_{\partial B_r(a)} \frac{f(w) - f_n(w)}{(w-z)^2} dw\right| \le \frac{1}{2\pi} 2\pi r * \frac{1}{\left(\frac{r}{2}\right)^2} \sup_{|w-a|=r} |f(w) - f_n(w)|$$

Which goes to 0 uniformly in z by local uniform convergence.

Proposition: If a sequence of holomorphic functions converges uniformly on a domain D then it converges uniformly on every compact subset of D.

Proof: Let K be a compact subset of D. For each point z in K, let U be an open ball around z such that our sequence converges uniformly, which is possible by local uniform convergence. We can do this for every z in K then take a finite subcover by compactness. Now ennumerate this finite subcover $U_1, U_2, \dots, U_k$. Now by definition of uniform convergence, for all ε>0, there is some $N_1, N_2, \dots, N_k$ such that for all $n > N_j$ and $w \in U_j$, $|f_n(w) - f(w)| < \varepsilon$. Now by taking $N = \max(N_1, N_2, \dots, N_k)$ we deduce that we have uniform convergence over all the U_j's, thus uniform convergence on D.

Proposition: If a sequence of injective holomorphic functions converges locally uniformly to f on a domain u then f is either injective or constant

Proof: Suppose f is non-constant and that $f(z_1) = f(z_2) = a$. Now as we have seen before these points are joined by a polygonal path in U, and it is possible to find a path that bounds a domain in U that contain $z_1$and z_2, just by openness.

By the identity theorem, f would be constant if it was a everywhere on this path. So the pre-image of a consists of isolated points and our path could dodge them. By the argument principle,

$$1 \ge \int_\gamma \frac{f_n{}'(z)}{f_n(z) - a} dz$$

Since f_n is injective for every n. A path is compact so f_n converges uniformly on this path by the previous proposition. So the integral converges to $\int_\gamma \frac{f'(z)}{f(z)-a}dz$ which is at least 2 by the argument principle which is a contradiction.

Definition: Let U be a domain and let F be a family of holomorphic functions on U. For the next few definitions, we have the same conditions. We say U is normal if every sequence of functions in F has a subsequence of functions that converges locally uniformly on U.

Definition: We say F is uniformly bounded on compact subsets of U if for every compact subset K of U there exists an M such that $|f(z)| \leq M$ for all z in K and f in F.

Definition: We say F is equicontinuous on a compact set K if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever z and w are in K and $|z - w| < \delta$, for all f in F we have $|f(z) - f(w)| < \varepsilon$. Note that we have seen this definition in level 6.

This condition basically means that we have uniform continuity uniformly on the family.

Definition: Let K_l be a countably infinite sequence of compact subsets of U. We say it is an exhaustion if 1. Each one is contained in the interior of the next one and 2. Their union equals U.

Lemma: Any domain $U \in \mathbb{C}$, in other words any open subset of $\mathbb{C}$, has an exhaustion.

Proof: We have seen something similar in the "Analysis lemmas" document. The idea is to let K_l be the set of all points in U such that their distance is $\geq \frac{1}{l}$ from the boundary of U, intersected with the compact set $|z| \leq l$. If two of these sets are the same we just take a subsequence such that they are not, as we certainly need infinitely many since they are all compact so finite unions of them will also be compact.

Theorem (Montel's Theorem): Every family F which is uniformly bounded is normal and equicontinuous

Proof:

Before we do it, note that it is not like the situation in the real numbers, where $f_n(x) = \sin(nx)$ on (0,1) is uniformly bounded but not equicontinuous or normal.

Let K be a compact subset of U and let r be small enough that $B_{3r}(z)$ is contained in U for all z in K, this is possible by the extreme value theorem as the "shortest distance to boundary function" is continuous on K and attains a minimum on K by the extreme value theorem as K is compact, and this minimum is not 0 since K is in U and U is open and thus U does not contain any of its boundary points and thus neither does K so the function cannot be 0 anywhere in K as it is only 0 on boundary points of U.

Let z,w be some 2 points in K with $|z - w| < r$ and let Y be the boundadry of the circle $B_{2r}(w)$, possible by how we chose K. Then we have $f(z) - f(w) = \frac{1}{2\pi i}\int_\gamma f(t)\left[\frac{1}{t-z} - \frac{1}{t-w}\right]dt$ by the cauchy integral formula.

Now $|f(z) - f(w)| \leq \frac{1}{2\pi}\int_\gamma |f(t)|\left|\frac{1}{t-z} - \frac{1}{t-w}\right|dt = \frac{1}{2\pi}\int_\gamma |f(t)|\frac{|z-w|}{|t-z||t-w|}dt \leq \frac{1}{2\pi r^2}\int_\gamma |f(t)||z-w|dt$

But now,

$$|f(z) - f(w)| \leq \frac{2\pi r}{2\pi r^2} * M * |z - w| = \frac{M}{r}|z - w|$$

Existence of this M follows from uniform boundedness applied to the compact set consisting of all points in U at a distance at most 2r from K. Therefore there is a C such that for EVERY z and w in K at most a distance R apart, $|f(z) - f(w)| < C|z - w|$. Thus equicontinuity on K follows.

We now just need the proof of normality. This will be identical to something which we proved level 6, so therefore I will not do it again.

Lemma: Let f be a holomorphic function with domain equal to $B_1(0)$ and image in $B_1(0)$ and suppose that $f(0) = 0$, then $f'(0) \leq 1$ and if f is not a rotation then $f'(0) < 1$.

Proof:

Since $f(0) = 0$, $\frac{f(z)}{z}$ can be defined with just a removable singularity. We know that $\left|\frac{f(z)}{z}\right| \leq \frac{1}{r}$ where $r = |z|$ for each z by assumption. By the local maximum principle, we conclude that this stays true for every z with $|z| \leq r$. So we can let r approach 1 in the limit and we conclude that $|f(z)| \leq z$ for all z. Now note that by definition of the derivative, if $g(z) = \frac{f(z)}{z}$ with the removable singularity removed then $g(0) = f'(0)$ so we get $f'(0) \leq 1$. Furthermore, if $f'(0) = 1$ then g is constant by the local maximum principle so f is indeed a rotation.

Lemma: Let U be a domain and let f be holomorphic on U. Let z_0 be in U and let D be some closed disc in U centered about z_0 with radius R. Then the n'th derivative of f at z_0 is bounded above as follows: $\left|f^{(n)}(z_0)\right| \leq \frac{n! \sup_{z \in \partial D}|f(z)|}{R^n}$

Proof:

By the cauchy integral formula,

$$\left|f^{(n)}(z_0)\right| = \left|\frac{n!}{2\pi i}\int_{\partial D}\frac{f(z)}{(z - z_0)^{n+1}}dz\right| = \frac{n!}{2\pi}\left|\int_0^{2\pi}\frac{f\left(z_0 + Re^{i\theta}\right)}{(Re^{i\theta})^{n+1}}Rie^{i\theta}d\theta\right| = \frac{n!}{2\pi}2\pi R\frac{\sup_{z \in \partial D}|f(z)|}{R^{n+1}}$$

So the result immediately follows by simplification.

We will now prove the riemann mapping theorem.

Theorem (Riemann mapping theorem): Let U∈ℂ be a simply connected domain and suppose that U is not ℂ. Then there is a conformal equivalence between U and $B_1(0)$

Proof:

This will be kind of long, but we will do it.

We will first prove that U is conformally equivalent to some subset of $B_1(0)$ containing 0. To do this, pick a complex number a that is NOT in U (since by hypothesis, this exists), which allows us to define $f(z) = \log(z - a)$ holomorphic everywhere on U. We have $e^{f(z)} = z - a$ which shows that f is injective (since $f(z) = f(w)$ implies $z - a = w - a$ so $z = w$).

Now pick a point w in U and observe that $f(z) \neq f(w) + 2\pi in$ for any z in U or non-zero integer n because otherwise we could exponentiate both sides to find that $z = w$ and therefore $f(z) = f(w)$.

Furthermore, there is a disc of points around $2\pi i + f(w)$ that f does not hit, because otherwise there exists a sequence z_n in U such that $f(z_n) \to f(w) + 2\pi i$ by continuity of f. But exponentiating this relation we find that $z_n \to w$ so by continuity $f(z_n) \to f(w)$ which is still a contradiction.

Now by our analysis, the map $F(z) = \frac{1}{f(z)-(f(w)+2\pi i)}$ is bounded and injective. It follows that F is conformal by our results around local mapping degree, since if F had a zero derivative anywhere then it would not be injective, since in some neighbourhood around that point there would be a value that F hits more than once. By moving and scaling the image of F appropriately, we have a conformal map from U to an open subset of $B_1(0)$ containing 0. We therefore may now assume that U is such a subset for the remainder of the proof. Note that U is also simply connected (every loop can be shrunk to a point by considering the pre-image of that loop on the starting U).

Let F be the family of injective functions from U to $B_1(0)$ that send 0 to 0. This is non-empty since it contains the identity and is uniformly bounded (since it is bounded above in absolute value by 1). By the previous lemma applied to a small disc centered at the origin, there is a uniform upper bound for $|f'(0)|$, so let s be the supremum of $f'(0)$ over F.

Now by the definition of supremum, there exists a sequence f_n in f such that $|f_n'(0)| \to s$ as $n \to \infty$. By montel's theorem, this has a subsequence that converges uniformly on compact sets to a holomorphic function f on U. By a previous proposition, the resulting function f is also injective, as since the identity function is in F, $s \geq 1$ so f is not constant. By the local maximum principle, we see that everywhere on U, $|f(z)| < 1$, since $|f(z)| \leq 1$ as it is a limit of such functions. Since $f(0) = 0$, it follows that this function f is actually in F, and that it attains $|f'(0)| = s$.

This f is actually the conformal map we are looking for, we just need to show that it is surjective.

We will do this by contradiction: We will show that if it were NOT surjective, we could find a function in F with $|f'(0)| > s$.

Suppose a is in the unit disc such that $f(z) \neq a$ for all z in U. Then consider the function

$$\psi(z) = \frac{a - z}{1 - \bar{a}z}$$

It is easy to check that this swaps 0 and a and that it sends the unit disc to itself. Specifically, for the last part, because if $|z| = 1$ then $|\psi(z)| = \frac{|a-z|}{|1-\bar{a}z|} = \frac{|a\bar{z}-1|}{|1-\bar{a}z|} = 1$ so it sends the boundary of the disc to itself, and it sends the unit disc to some simply connected domain, and if there were some exterior point in the image then as you move from the corresponding point in the pre-image to 0 you will cross the boundary contradicting the fact that this is a mobius map and hense is injective, so every point in the image is in the unit disc, and moreover every point in the unit disc has a preimage in the unit disc since mobius transformations are invertible, so it is a surjection onto the unit disc.

Since U is simply connected and $\psi(f(U))$ is also a conformal image of U, $\psi(f(U))$ is also simply connected for the same reason that U was, and $\psi(f(U))$ does not contain the origin. It is therefore possible to define a function g on $\psi(f(U))$ by $g(w) = e^{\frac{1}{2}\log(w)}$.

Now define $\phi(z) = \frac{g(a)-z}{1-\overline{g(a)}z}$. Then let $H(z) := \phi\left(g\left(\psi(f(z))\right)\right)$. Now observe:

- H is holomorphic

- H(0) is $\phi\left(g\big(\psi(0)\big)\right) = \phi\big(g(a)\big) = 0$
- H maps into $B_1(0)$ since it is true for each of the functions in the composition
- H is injective because it is the composition of 4 injective functions

Thus H is in F.

Now define $G(z) = \psi^{-1}(\phi^{-1}(z)^2)$, which is a non-injective function from the unit disc to the unit disc. It is non-injective since it has the squared in the middle. Now we note that by construction,

$$f(z) = G\big(H(z)\big)$$

But G maps D into D with G(0)=0 and it is not injective so not a rotation so it follows that $G'(0) < 1$. But by the chain rule, $f'(0) = G'\big(H(0)\big)H'(0) = G'(0)H'(0)$, so $|f'(0)| < |H'(0)|$, which is the contradiction we wanted.